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Realizations of the q-Heisenberg and q-Virasoro Algebras

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Abstract

We give field theoretic realizations of both the q-Heisenberg and the q-Virasoro algebra. In particular, we obtain the operator product expansions among the current and the energy momentum tensor obtained using the Sugawara construction.

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Quantum algebras or more precisely quantized universal enveloping algebras first appeared in connection with the study of the inverse scattering problem [1]. Subsequently it was shown that these algebras are also deeply rooted in other areas such as exactly soluble statistical models [2], factorizable S-matrix theory [3] and conformal field theory [4]. Mathematically, these algebras are Hopf algebras which are non cocommutative. These can be compared with the classical universal enveloping algebras which can be endowed with cocommutative Hopf structures. In this regard quantum algebras appear as natural generalizations of the usual Lie algebras.

Lately there has been a lot of interest in the q deformation of the Virasoro algebra [6-9]. By generalizing a differential realization of $\mathrm{su}_q(1,1)$, Curtright and Zachos (CZ) [6] obtained a q-analoque of the centerless Virasoro algebra. Its central extension was later furnished by Aizawa and Sato [7]. However, to date, the existence of a Hopf structure for this algebra remains an open question.

More recently Chaichian and Prešnajder [8] proposed a different version of the q-Virasoro algebra by carrying out a Sugawara construction on a q-analogue of an infinite dimensional Heisenberg algebra ($\mathrm{H}_q(\infty)$). They also showed that in a unitary representation, this algebra possesses a primitive (cocommutative) Hopf structure.

In this paper we realize both the q-Heisenberg as well as the q-Virasoro algebra using field operators. In particular, we obtain the operator product expansions (OPE's) of these field operators using some of the standard techniques of conformal field theory. The central term for the q-Virasoro algebra obtained via the realization is shown to differ slightly from the one given in ref.[8]. It is further shown that our expression leads to the standard case in the $q \to 1$ limit. †

We begin by summarizing some results of ref.[8] that will be used later. The algebra of $\mathrm{H}_q(\infty)$ is based on the one dimensional Heisenberg algebra which is a bona fide Hopf algebra with relations [10] ‡

$$\begin{aligned} &[a, a^\dagger] \equiv a a^\dagger - a^\dagger a = \frac{\sinh(\epsilon H/2)}{\epsilon/2} \\ &[H, a] = 0, \qquad [H, a^\dagger] = 0 \end{aligned} \tag{1}$$

† Contrary to the claims of ref.[8], the central term presented there does not have the correct limit.

‡ It is worth noting that this algebra differs from those given in refs.[11].

with $q = e^{\epsilon}$ as the deformation parameter. Here a and $a^{\dagger}$ can be regarded as the annhilation and creation operators repectively. The algebra of $\mathrm{H}_q(\infty)$ is obtained by considering an infinite collection of these operators labelled as

$$a_n, \quad a_{-n} = a_n^{\dagger} \qquad n = 1, 2, ...,$$

with commutation relations

$$\begin{aligned} [a_m, a_n] &= \omega_{mn} \\ [H, a_m] &= 0 \end{aligned} \tag{2}$$

where

$$\omega_{mn} = \frac{1}{\epsilon}\sinh(m\epsilon H)\delta_{m+n,0}. \tag{3}$$

The associative algebra generated by $\{\mathbf{1}, H, a_n\}_{n\in \mathrm{Z}}$ with the above relations can be endowed with Hopf structure by defining the following:

(i) co-product:

$$\begin{aligned} \Delta a_n &= q^{-|n|H/2} \otimes a_n + a_n \otimes q^{|n|H/2}, \\ \Delta H &= \mathbf{1} \otimes H + H \otimes \mathbf{1}, \quad \Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}; \end{aligned} \tag{4a}$$

(ii) co-unit:

$$\epsilon(a_n) = 0, \quad \epsilon(H) = 0, \quad \epsilon(\mathbf{1}) = \mathbf{1}; \tag{4b}$$

(iii) antipode:

$$S(a_n) = -a_n, \quad S(H) = -H, \quad S(\mathbf{1}) = \mathbf{1}. \tag{4c}$$

The q-deformed Virasoro generators in the Sugawara construction read as

$$L_m^{\alpha} = \frac{1}{2}\sum_{k,n} \cosh(\frac{k-n}{2}\epsilon\alpha H) : a_k a_n : \delta_{k+n,m}. \tag{5}$$

The normal ordering prescription here is taken as

$$: a_k a_n := a_k a_n - \theta(k)\omega_{kn} \tag{6}$$

where ω_{kn} is defined in (3) and $\theta(k) = 1$ or 0 for k positive or negative respectively. It is worth noting that q-Virasoro generator carries an additional integer-valued index which is

required for the commutators between the generators to close. The commutation relations furnished in ref.[8] are given by

$$
\begin{aligned}
[L_m^\alpha, L_n^\beta] = & \frac{1}{2\epsilon}\sinh(\frac{m-n-n\alpha+m\beta}{2}\epsilon H)L_{m+n}^{\alpha+\beta+1} \\
& + \frac{1}{2\epsilon}\sinh(\frac{m-n+n\alpha-m\beta}{2}\epsilon H)L_{m+n}^{-\alpha-\beta+1} \\
& + \frac{1}{2\epsilon}\sinh(\frac{m-n+n\alpha+m\beta}{2}\epsilon H)L_{m+n}^{\alpha-\beta-1} \\
& + \frac{1}{2\epsilon}\sinh(\frac{m-n-n\alpha-m\beta}{2}\epsilon H)L_{m+n}^{-\alpha+\beta-1} \\
& + \frac{1}{16\epsilon^2}(C_{m-1}^{\alpha,\beta} + C_{m-1}^{\alpha,-\beta})\delta_{m+n,0},
\end{aligned} \tag{7a}
$$

with

$$
\begin{aligned}
C_m^{\alpha,\beta} = & \frac{\sinh(\frac{\alpha+\beta+1}{2}m\epsilon H)}{\sinh(\frac{\alpha+\beta+1}{2}\epsilon H)} - 2\cosh(m\epsilon H + \epsilon H)\frac{\sinh(\frac{\alpha+\beta}{2}m\epsilon H)}{\sinh(\frac{\alpha+\beta}{2}\epsilon H)} \\
& + \frac{\sinh(\frac{\alpha+\beta-1}{2}m\epsilon H)}{\sinh(\frac{\alpha+\beta-1}{2}\epsilon H)}.
\end{aligned} \tag{7b}
$$

Here a few remarks are in order. Firstly the $q \to 1$ limit yields

$$
[L_m^\alpha, L_n^\beta] \to [L_m, L_n] = (m-n)HL_{m+n} - \frac{1}{96}m(m-1)(11m+2)H^2\delta_{m+n,0} \tag{8}
$$

which shows that although the operator part i.e. terms involving the generators reduce to the usual expression, the central term does not. Secondly the Hopf structure that can be written down for the algebra is obvious only for the case when H is a constant or when a unitary irreducible representation is chosen. In this case, the algebra becomes an infinite dimensional Lie algebra and this can be endowed with a primitive (cocommutative) Hopf structure. In general however, with H non-trivial in (7), the existence of a Hopf structure has not be shown. Even if it does exist, as pointed out in ref.[8], it would most likely be non-trivial and complicated.

To give a field-theoretic realization of an algebra, one must essentially furnish the operator product expansion (OPE) between the appropriate field operators. In fact only the singular part of this expansion is essential as it embodies all the relevant information about the algebra. For instance, the usual Heisenberg algebra can be obtained from the singular portion of the OPE between two currents:

$$
J(z)J(w) = \frac{1}{(z-w)^2} + \text{regular terms}. \tag{9}
$$

To obtain a q-analogue of this OPE, we begin by defining the current in the usual way,

$$J(z) = \sum_{m=-\infty}^{\infty} a_m z^{-m-1} \tag{10}$$

with the operators $\{a_m\}$ satisfying (2) instead of the usual Heisenberg algebra. Here we will restrict ourselves to the case of a unitary representation in which $H = 1$. To simplify the notation we write

$$[a_m, a_n] = \kappa\ [m]\ \delta_{m+n,0} \tag{11}$$

where

$$[x] \equiv \frac{q^m - q^{-m}}{q - q^{-1}} \qquad \text{and} \qquad \kappa = \frac{1}{\epsilon}\sinh(\epsilon).$$

Then by using (10) and (11) we have for $|w| < |z|$,

$$\begin{aligned} J(z)J(w) &= \sum_{m,n} a_m a_n z^{-m-1} w^{-n-1} \\ &=: J(z)J(w) : + \frac{1}{zw} \sum_{m>0} \kappa\ [m]\ (\frac{w}{z})^m \\ &=: J(z)J(w) : + \frac{\kappa}{(z-w)_q^2} \end{aligned} \tag{12}$$

where $(z-w)_q^2 = (z - wq^{-1})(z - wq)$. It is interesting to note that the poles are located at two points, $\{wq^{-1}, wq\}$ and both are of order 1. This differs from the standard case where there is a single pole of order 2 at $z = w$. However, in the $q \to 1$ limit these poles coalesce to form a pole of order 2 and expression (12) reduces to (9). The situation here is quite similar to that of refs.[7] and [12] where a realization for the CZ algebra [6] also leads to such degeneracies in the poles.

Before proceeding further, it is instructive to check whether the above q-OPE leads to the q-Heisenberg algebra as required. To this end, we first invert (10) to give

$$a_m = \oint_C \frac{dz}{2\pi i} J(z) z^m \tag{13}$$

where the contour C is taken as $|z| =$ constant. Here we use the usual prescription of radial quantization of standard conformal field theory in which different 'times' correspond to concentric circles of different radii. In this context, time-ordering is replaced by that of radial-ordering:

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w|. \end{cases} \tag{14}$$

Then by using the standard procedure for computing an 'equal-time' commutator [13], we have

$$\begin{aligned}
[a_m, a_n] &= [\oint_C \frac{dz}{2\pi i} J(z) z^m, \oint_C \frac{dw}{2\pi i} J(w) w^n] \\
&= \oint_C \frac{dw}{2\pi i} (\oint_{|z|>|w|} - \oint_{|z|<|w|}) \frac{dz}{2\pi i} z^m w^n R(J(z)J(w)) \\
&= \oint_C \frac{dw}{2\pi i} \oint_{C_P} \frac{dz}{2\pi i} z^m w^n R(J(z)J(w))
\end{aligned} \tag{15}$$

where the integral over z is taken around all the poles in the OPE of $J(z)J(w)$. Now the above procedure only makes sense if we assume that the singularities of $J(z)J(w)$ are located on the $|z| = |w|$ contour since otherwise these poles will not make any contribution to the integral. For the q-OPE (12) this requires that q be a pure phase (or $|q| = 1$). Consequently by substituting (12) into (15) and using the fact

$$\oint_{C_P} \frac{dz}{2\pi i} \frac{z^m}{(z-w)_q^2} = \partial_z^q z^m|_{z=w} = [m] w^{m-1} \tag{16}$$

we obtain (11).

In analogy to the standard case, we now define the energy-momentum tensor as

$$T^\alpha(z) = \sum_m L_m^\alpha z^{-m-2} \tag{17}$$

where the index α also appears on T by virtue of its presence on L.

To be consistent with (5), the corresponding Sugawara construction for $T^\alpha(z)$ in terms of the currents reads as

$$T^\alpha(z) = \frac{1}{4} : J(zq^{\alpha/2}) J(zq^{-\alpha/2}) : + \frac{1}{4} : J(zq^{-\alpha/2}) J(zq^{\alpha/2}) : . \tag{18}$$

It is worth noting that the second term on the right hand side is identical to the first with q and q^{-1} interchanged. (In the following such terms will be denoted by $q \leftrightarrow q^{-1}$ for simplicity.) It is easy to verify that (10) and (17) together with (18) lead to (5).

Next let us examine the OPE between $T^\alpha(z)$ and $J(w)$:

$$T^\alpha(z) J(w) = \frac{1}{4} : J(zq^{\alpha/2}) J(zq^{-\alpha/2}) : J(w) \; + \; q \leftrightarrow q^{-1}. \tag{19}$$

Since only the singular part is of interest, we have using (12)

$$\begin{aligned}
T^\alpha(z) J(w) &\sim \frac{1}{2} < J(zq^{\alpha/2}) J(w) > J(zq^{-\alpha/2}) \; + \; q \leftrightarrow q^{-1} \\
&\sim \frac{1}{2} \frac{\kappa}{(zq^{\alpha/2} - w)_q^2} J(zq^{-\alpha/2}) + q \leftrightarrow q^{-1}.
\end{aligned} \tag{20}$$

Then by expanding the field $J(zq^{-\alpha/2})$ using the q-Taylor's series †: (see ref.[7])

$$J(zq^{-\alpha/2}) = J(wq^{-\alpha-1}) + (zq^{-\alpha/2} - wq^{-\alpha-1})\partial_w^q J(wq^{-\alpha}) + ... \tag{21}$$

the above OPE reduces to

$$T^\alpha(z)J(w) \sim \frac{\kappa}{2}\{\frac{J(wq^{-\alpha-1})}{(zq^{\alpha/2}-w)_q^2} \; + \; \frac{q^{-\alpha}\partial_w^q J(wq^{-\alpha})}{(zq^{\alpha/2}-wq)}\} \; + \; q \leftrightarrow q^{-1} \tag{22}$$

which is the singular part of the q-OPE between $T^\alpha(z)$ and $J(w)$.

It is again instructive to compare the commutator between L_m^α and a_n obtained from the q-OPE above with that evaluated directly from the commutation relations. To this end we have after a short computation

$$\begin{aligned}[L_m^\alpha, a_n] &= \oint_C \frac{dw}{2\pi i}\oint_{C_P}\frac{dz}{2\pi i} z^{m+1} w^n T^\alpha(z)J(w) \\ &= -\kappa \cosh(\alpha(2n+m)\epsilon/2)[n]a_{m+n}\end{aligned} \tag{23}$$

which is precisely what one would obtain if the bracket was evaluated directly from the definition (5) and the relations (11).

With the q-OPE between $T^\alpha(z)$ and $J(w)$ so obtained we can go on to compute the q-OPE between $T^\alpha(z)$ and $T^\beta(w)$. Indeed, by writing

$$T^\alpha(z)T^\beta(w) = \lim_{w'\to w}\frac{1}{4}T^\alpha(z) : J(wq^{\beta/2})J(w'q^{-\beta/2}) : \;\; + \;\; q \leftrightarrow q^{-1} \tag{24}$$

and using (22) we have after a lengthy calculation

$$\begin{aligned}T^\alpha(z)T^\beta(w) \sim \frac{\kappa}{2(q-q^{-1})w}\{&\frac{T^{\alpha+\beta+1}(wq^{(\alpha+1)/2})}{(zq^{-(\alpha-\beta)/2}-wq^{\beta+1})} + \frac{T^{-\alpha+\beta-1}(wq^{(\alpha+1)/2})}{(zq^{-(\alpha+\beta)/2}-wq^{-\beta+1})} \\ &-\frac{T^{-\alpha-\beta+1}(wq^{(\alpha-1)/2})}{(zq^{-(\alpha-\beta)/2}-wq^{\beta-1})} - \frac{T^{\alpha-\beta-1}(wq^{(\alpha-1)/2})}{(zq^{-(\alpha+\beta)/2}-wq^{-\beta-1})}\} \\ +\frac{\kappa^2}{4(q-q^{-1})w^3}\{&\frac{1}{(q^{\alpha+\beta/2+1}-q^{-\beta/2})_q^2}\frac{1}{(zq^{-(\alpha-\beta)/2}-wq^{\beta+1})} \\ &+\frac{1}{(q^{\alpha-\beta/2+1}-q^{\beta/2})_q^2}\frac{1}{(zq^{-(\alpha+\beta)/2}-wq^{-\beta+1})} \\ &-\frac{1}{(q^{\alpha+\beta/2-1}-q^{-\beta/2})_q^2}\frac{1}{(zq^{-(\alpha-\beta)/2}-wq^{\beta-1})} \\ &-\frac{1}{(q^{\alpha-\beta/2-1}-q^{\beta/2})_q^2}\frac{1}{(zq^{-(\alpha+\beta)/2}-wq^{-\beta-1})}\} \\ &+ \qquad q \leftrightarrow q^{-1}\end{aligned} \tag{25}$$

† In (21) we only retain the first two terms of the expansion as the rest contain the factor $(zq^{-\alpha/2}-w)_q^2$ which cancels with the term in the denominator of (20).

where the terms in the first bracket correspond to the operator part while those in the second are the anomaly terms. From this q-OPE we can obtain the q-Virasoro algebra by evaluating the integrals in

$$[L_m^\alpha, L_n^\beta] = \oint_C \frac{dw}{2\pi i} \oint_{C_P} \frac{dz}{2\pi i} z^{m+1} w^{n+1} T^\alpha(z) T^\beta(w). \tag{26}$$

For the operator part the terms are identical to those of (7a) but central term now yields

$$-\frac{1}{16\epsilon^2}({C'}_{m+1}^{\alpha,\beta} + {C'}_{m+1}^{\alpha,-\beta})\delta_{m+n,0}, \tag{27}$$

where

$$\begin{aligned} {C'}_m^{\alpha,\beta} = & \frac{\sinh(\frac{\alpha+\beta+2}{2}m\epsilon)}{\sinh(\frac{\alpha+\beta+2}{2}\epsilon)} - 2\cosh(m\epsilon - \epsilon)\frac{\sinh(\frac{\alpha+\beta}{2}m\epsilon)}{\sinh(\frac{\alpha+\beta}{2}\epsilon)} \\ & + \frac{\sinh(\frac{\alpha+\beta-2}{2}m\epsilon)}{\sinh(\frac{\alpha+\beta-2}{2}\epsilon)}. \end{aligned} \tag{28}$$

It is interesting to note that in the limit $q \to 1$ (or $\epsilon \to 0$) the central term reduces to

$$\frac{1}{12}m(m-1)(m+1)\delta_{m+n,0}$$

which is usual central term that one has for the Virasoro algebra.

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